Chapter 8. Dynamics II: Motion in a Plane

A roller coaster doing a loop-the-loop is a dramatic example of circular motion. But why doesn’t the car fall off the track when it’s upside down at the top of the loop? To answer this question, we must study how objects move in circles.

Chapter Goal: To learn how to solve problems about motion in a plane.
Chapter 8. Dynamics II: Motion in a Plane

Topics:

• Dynamics in Two Dimensions
• Velocity and Acceleration in Uniform Circular Motion
• Dynamics of Uniform Circular Motion
• Circular Orbits
• Fictitious Forces
• Why Does the Water Stay in the Bucket?
• Nonuniform Circular Motion
Chapter 8. Reading Quizzes
Circular motion is best analyzed in a coordinate system with

A. $x$- and $y$-axes.
B. $x$, $y$, and $z$-axes.
C. $x$- and $z$-axes.
D. $r$, $t$, and $z$-axes.
Circular motion is best analyzed in a coordinate system with

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B. $x$-, $y$-, and $z$-axes.
C. $x$- and $z$-axes.
D. $r$-, $t$-, and $z$-axes.

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This chapter studies

A. uniform circular motion.
B. nonuniform circular motion.
C. orbital motion.
D. Both a and b.
E. All of a, b, and c.
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A. uniform circular motion.
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For uniform circular motion, the net force

A. points toward the center of the circle.
B. points toward the outside of the circle.
C. is tangent to the circle.
D. is zero.
For uniform circular motion, the net force

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C. is tangent to the circle.
D. is zero.
Chapter 8. Basic Content and Examples
Dynamics in Two Dimensions

Suppose the $x$- and $y$-components of acceleration are independent of each other. That is, $a_x$ does not depend on $y$ or $v_y$, and $a_y$ does not depend on $x$ or $v_x$. Your problem-solving strategy is to

1. draw a pictorial representation — a motion diagram (if needed) and a free-body diagram, and
2. use Newton’s second law in component form.

$$(F_{\text{net}})_x = \sum F_x = ma_x \quad \text{and} \quad (F_{\text{net}})_y = \sum F_y = ma_y$$

The force components (including proper signs) are found from the free-body diagram.
Dynamics in Two Dimensions

3. Solve for the acceleration. If the acceleration is constant, use the two-dimensional kinematic equations of Chapter 4 to find velocities and positions.

\[ x_f = x_i + v_{ix} \Delta t + \frac{1}{2} a_x (\Delta t)^2 \]
\[ y_f = y_i + v_{iy} \Delta t + \frac{1}{2} a_y (\Delta t)^2 \]
\[ v_{fx} = v_{ix} + a_x \Delta t \]
\[ v_{fy} = v_{iy} + a_y \Delta t \]
Projectile Motion

In the absence of air resistance, a projectile has only one force acting on it: the gravitational force, \( F_G = mg \), in the downward direction. If we choose a coordinate system with a vertical \( y \)-axis, then

\[
\vec{F}_G = -mg \hat{j}
\]

Consequently, from Newton’s second law, the acceleration is

\[
\begin{align*}
a_x &= \frac{(F_G)_x}{m} = 0 \\
a_y &= \frac{(F_G)_y}{m} = -g
\end{align*}
\]

The vertical motion is free fall, while the horizontal motion is one of constant velocity.
Uniform Circular Motion

The acceleration of uniform circular motion points to the center of the circle. Thus the acceleration vector has only a radial component \( a_r \). This acceleration is conveniently written in the \( rtz \)-coordinate system as

\[
a_r = \frac{v^2}{r} = \omega^2 r
\]

\[
a_t = 0
\]

\[
a_z = 0
\]
**EXAMPLE 8.2 The acceleration of an atomic electron**

**QUESTION:**

We will later study the Bohr atom. This is a simple model of the hydrogen atom in which an electron orbits a proton at a radius of $5.29 \times 10^{-11}$ m with a period of $1.52 \times 10^{-16}$ s. What is the electron’s centripetal acceleration?
EXAMPLE 8.2 The acceleration of an atomic electron

**SOLVE** From Chapter 4, the electron’s speed is

\[ v = \frac{2\pi r}{T} = \frac{2\pi (5.29 \times 10^{-11} \text{ m})}{1.52 \times 10^{-16} \text{ s}} = 2.19 \times 10^6 \text{ m/s} \]

Then from Equations 8.7,

\[ a_r = \frac{v^2}{r} = \frac{(2.19 \times 10^6 \text{ m/s})^2}{5.29 \times 10^{-11} \text{ m}} = 9.07 \times 10^{22} \text{ m/s}^2 \]
EXAMPLE 8.2 The acceleration of an atomic electron

ASSESS This example demonstrates the unbelievably enormous accelerations that take place at the atomic level. It should then come as no surprise that atomic particles may behave in ways that our intuition, trained by accelerations of only a few m/s², cannot easily grasp.
The usefulness of the **rtz**-coordinate system becomes apparent when we write Newton’s second law in terms of the *r*-, *t*-, and *z*-components, as follows:

\[
(F_{\text{net}})_r = \sum F_r = ma_r = \frac{mv^2}{r} = m\omega^2 r
\]

\[
(F_{\text{net}})_t = \sum F_t = ma_t = 0
\]

\[
(F_{\text{net}})_z = \sum F_z = ma_z = 0
\]
QUESTION:

What is the maximum speed with which a 1500 kg car can make a left turn around a curve of radius 50 m on a level (unbanked) road without sliding?
EXAMPLE 8.4 Turning the corner I

**MODEL** Although the car turns only a quarter of a circle, we can model the car as a particle in uniform circular motion as it goes around the turn. Assume that rolling friction is negligible.
EXAMPLE 8.4 Turning the corner

**VISUALIZE FIGURE 8.8** shows the pictorial representation. The car moves along a circular arc at constant speed for the quarter-circle necessary to complete the turn.

**FIGURE 8.8** Pictorial representation of a car turning a corner.

Known
- $m = 1500 \text{ kg}$
- $r = 50 \text{ m}$
- $\mu_s = 1.0$

Find
- $v_{\text{max}}$

Top view of car
- $\vec{v}$
- $\vec{r}$

Top view of tire
- $\vec{f}_s$
- $\vec{f}_n$

This force prevents the tire from slipping sideways.

Rear view of car
- $\vec{F}_{\text{net}}$
- $\vec{F}_G$
- $\vec{n}$
EXAMPLE 8.4 Turning the corner I

Figure 8.8 shows the top view of a tire as it turns a corner. If the road surface were frictionless, the tire would slide straight ahead. The force that prevents an object from sliding across a surface is static friction. Static friction $\vec{f}_S$ pushes sideways on the tire, toward the center of the circle. How do we know the direction is sideways? If $\vec{f}_S$ had a component either parallel to $\vec{v}$ or opposite to $\vec{v}$, it would cause the car to speed up or slow down. Because the car changes direction but not speed, static friction must be perpendicular to $\vec{v}$. $\vec{f}_S$ causes the centripetal acceleration of circular motion around the curve, and thus the free-body diagram, drawn from behind the car, shows the static friction force pointing toward the center of the circle.
EXAMPLE 8.4 Turning the corner I

**SOLVE** Because the static friction force has a maximum value, there will be a maximum speed with which a car can turn without sliding. The maximum speed is reached when the static friction force reaches its maximum $f_{s\text{max}} = \mu_s n$. If the car enters the curve at a speed higher than the maximum, static friction will not be large enough to provide the necessary centripetal acceleration and the car will slide.
EXAMPLE 8.4 Turning the corner I

The static friction force points in the positive $r$-direction, so its radial component is simply the magnitude of the vector: $(f_s)_r = f_s$. Newton’s second law in the $rt_z$-coordinate system is

$$\sum F_r = f_s = \frac{mv^2}{r}$$

$$\sum F_z = n - mg = 0$$
EXAMPLE 8.4 Turning the corner I

The only difference from Example 8.3 is that the tension force toward the center has been replaced by a static friction force toward the center. From the radial equation, the speed is

$$v = \sqrt{\frac{rf_s}{m}}$$

The speed will be a maximum when $f_s$ reaches its maximum value:

$$f_s = f_{s\text{ max}} = \mu_s n = \mu_s mg$$

where we used $n = mg$ from the $z$-equation. At that point,

$$v_{\text{max}} = \sqrt{\frac{rf_{s\text{ max}}}{m}} = \sqrt{\mu_s rg}$$

$$= \sqrt{(1.0)(50 \, \text{m})(9.80 \, \text{m/s}^2)} = 22 \, \text{m/s}$$

where we found $\mu_s = 1.0$ in Table 6.1.
EXAMPLE 8.4 Turning the corner I

**ASSESS** 22 m/s ≈ 45 mph, a reasonable answer for how fast a car can take an unbanked curve. Notice that the car’s mass canceled out and that the final equation for $v_{\text{max}}$ is quite simple. This is another example of why it pays to work algebraically until the very end.
EXAMPLE 8.6 A rock in a sling

QUESTION:

A Stone Age hunter places a 1.0 kg rock in a sling and swings it in a horizontal circle around his head on a 1.0-m-long vine. If the vine breaks at a tension of 200 N, what is the maximum angular speed, in rpm, with which he can swing the rock?
EXAMPLE 8.6 A rock in a sling

MODEL  Model the rock as a particle in uniform circular motion.
EXAMPLE 8.6 A rock in a sling

**VISUALIZE** This problem appears, at first, to be essentially the same as Example 8.3, where the father spun his child around on a rope. However, the lack of a normal force from a supporting surface makes a big difference. In this case, the only contact force on the rock is the tension in the vine. Because the rock moves in a horizontal circle, you may be tempted to draw a free-body diagram like **FIGURE 8.11a**, where \( \vec{T} \) is directed along the \( r \)-axis. You will quickly run into trouble, however, because this diagram has a net force in the \( z \)-direction and it is impossible to satisfy \( \sum F_z = 0 \). The gravitational force \( \vec{F}_G \) certainly points vertically downward, so the difficulty must be with \( \vec{T} \).

**FIGURE 8.11** Pictorial representation of a rock in a sling.

<table>
<thead>
<tr>
<th>(a)</th>
<th>(b)</th>
</tr>
</thead>
</table>

- **Wrong diagram!**

<table>
<thead>
<tr>
<th>Center of circle</th>
</tr>
</thead>
<tbody>
<tr>
<td>( m = 1.0 \text{ kg} )</td>
</tr>
<tr>
<td>( L = 1.0 \text{ m} )</td>
</tr>
<tr>
<td>( T_{\text{max}} = 200 \text{ N} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Known</th>
</tr>
</thead>
<tbody>
<tr>
<td>( F_G )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Find</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega_{\text{max}} )</td>
</tr>
</tbody>
</table>
EXAMPLE 8.6 A rock in a sling

As an experiment, tie a small weight to a string, swing it over your head, and check the angle of the string. You will quickly discover that the string is not horizontal but, instead, is angled downward. The sketch of Figure 8.11b labels the angle \( \theta \). Notice that the rock moves in a horizontal circle, so the center of the circle is not at his hand. The \( r \)-axis points to the center of the circle, but the tension force is directed along the vine. Thus the correct free-body diagram is the one in Figure 8.11b.
EXAMPLE 8.6 A rock in a sling

**SOLVE** The free-body diagram shows that the downward gravitational force is balanced by an upward component of the tension, leaving the radial component of the tension to cause the centripetal acceleration. Newton's second law is

\[ \sum F_r = T \cos \theta = \frac{mv^2}{r} \]

\[ \sum F_z = T \sin \theta - mg = 0 \]

where \( \theta \) is the angle of the vine below horizontal.
EXAMPLE 8.6 A rock in a sling

From the \( z \)- equation we find

\[
\sin \theta = \frac{mg}{T}
\]

\[
\theta = \sin^{-1}\left(\frac{(1.0 \, \text{kg})(9.8 \, \text{m/s}^2)}{200 \, \text{N}}\right) = 2.81^\circ
\]

where we’ve evaluated the angle at the maximum tension of 200 N. The vine’s angle of inclination is small but not zero.
EXAMPLE 8.6 A rock in a sling

Turning now to the $r$-equation, we find the rock’s speed is

$$v = \sqrt{\frac{rT \cos \theta}{m}}$$

Careful! The radius $r$ of the circle is not the length $L$ of the vine. You can see in Figure 8.11b that $r = L \cos \theta$. Thus

$$v = \sqrt{\frac{LT \cos^2 \theta}{m}} = \sqrt{\frac{(1.0 \text{ m})(200 \text{ N})(\cos 2.81^\circ)^2}{1.0 \text{ kg}}} = 14.1 \text{ m/s}$$

We can now find the maximum angular speed, the value of $\omega$ that brings the tension to the breaking point:

$$\omega_{\text{max}} = \frac{v}{r} = \frac{v}{L \cos \theta} = \frac{14.1 \text{ rad}}{1 \text{ s}} \times \frac{1 \text{ rev}}{2\pi \text{ rad}} \times \frac{60 \text{ s}}{1 \text{ min}} = 135 \text{ rpm}$$
Circular Orbits

**FIGURE 8.13** The “real” gravitational force is always directed toward the center of the planet.

(a) Projectile

Parabolic trajectory

Flat-earth approximation

(b) Projectile

Circular orbit

Spherical planet
Circular Orbits

An object moving in a circular orbit of radius $r$ at speed $v_{\text{orbit}}$ will have centripetal acceleration of

$$a_r = \frac{(v_{\text{orbit}})^2}{r} = g$$

That is, if an object moves parallel to the surface with the speed

$$v_{\text{orbit}} = \sqrt{rg}$$

then the free-fall acceleration provides exactly the centripetal acceleration needed for a circular orbit of radius $r$. An object with any other speed will not follow a circular orbit.
Fictitious Forces

• If you are riding in a car that makes a sudden stop, you may feel as if a force “throws” you forward toward the windshield.
• There really is no such force.
• Nonetheless, the fact that you seem to be hurled forward relative to the car is a very real experience!
• You can describe your experience in terms of what are called fictitious forces.
• These are not real forces because no agent is exerting them, but they describe your motion relative to a noninertial reference frame.
FIGURE 8.14 The forces are properly identified only in an inertial reference frame.

The car is at rest in the passenger’s frame.

Force of being thrown into the windshield. This is a fictitious force.

Noninertial reference frame of passenger
The car is decelerating.

The passenger continues forward with constant velocity.

Inertial reference frame of the ground
Centrifugal Force

If the car you are in turns a corner quickly, you feel “thrown” against the door. The “force” that seems to push an object to the outside of a circle is called the centrifugal force. It describes your experience relative to a noninertial reference frame, but there really is no such force.
Why Does the Water Stay in the Bucket?

• Imagine swinging a bucket of water over your head. If you swing the bucket quickly, the water stays in. But you’ll get a shower if you swing too slowly.
• The critical angular velocity $\omega_c$ is that at which gravity alone is sufficient to cause circular motion at the top.

$$\omega_c = \sqrt{\frac{g}{r}}$$
Why Does the Water Stay in the Bucket?

**FIGURE 8.18** A roller coaster car at the top of the loop.

The normal force adds to gravity to make a large enough force for the car to turn the circle.

\[ v > v_c \]
Why Does the Water Stay in the Bucket?

At $v_c$, gravity alone is enough force for the car to turn the circle. $\vec{n} = \vec{0}$ at the top point.
**Why Does the Water Stay in the Bucket?**

If \( \omega < \omega_c \), the gravitational force is too large. It pulls the water out of the circle and into a tighter parabolic trajectory.

\[
\vec{F}_G
\]

Parabolic trajectory

Normal force became zero here.
Nonuniform Circular Motion

**FIGURE 8.21** Net force $\vec{F}_{\text{net}}$ is applied to a particle moving in a circle.

The radial force causes the centripetal acceleration $a_r$.

The tangential force causes the tangential acceleration $a_t$. 
Nonuniform Circular Motion

- The force component \((F_{\text{net}})_r\) creates a centripetal acceleration and causes the particle to change directions.
- The component \((F_{\text{net}})_t\) creates a tangential acceleration and causes the particle to change speed.
- Force and acceleration are related to each other through Newton’s second law.

\[
(F_{\text{net}})_r = \sum F_r = ma_r = \frac{mv^2}{r} = m\omega^2 r
\]

\[
(F_{\text{net}})_t = \sum F_t = ma_t
\]

\[
(F_{\text{net}})_z = \sum F_z = 0
\]
Problem-Solving Strategy: Circular-Motion Problems

**PROBLEM-SOLVING STRATEGY 8.1 Circular-motion problems**

**MODEL** Make simplifying assumptions.
Problem-Solving Strategy: Circular-Motion Problems

**VISUALIZE** Draw a pictorial representation.

- Establish a coordinate system with the r-axis pointing toward the center of the circle.
- Show important points in the motion on a sketch. Define symbols and identify what the problem is trying to find.
- Identify the forces and show them on a free-body diagram.
Problem-Solving Strategy: Circular-Motion Problems

SOLVE Newton’s second law is

\[(F_{\text{net}})_r = \sum F_r = ma_r = \frac{mv^2}{r} = m\omega^2 r\]

\[(F_{\text{net}})_t = \sum F_t = ma_t\]

\[(F_{\text{net}})_z = \sum F_z = 0\]

- Determine the force components from the free-body diagram. Be careful with signs.
- Solve for the acceleration, then use kinematics to find velocities and positions.
**Problem-Solving Strategy: Circular-Motion Problems**

**ASSESS** Check that your result has the correct units, is reasonable, and answers the question.
Chapter 8. Summary Slides
Newton’s Second Law

Expressed in $x$- and $y$-component form:

$$(F_{\text{net}})_x = \sum F_x = ma_x$$

$$(F_{\text{net}})_y = \sum F_y = ma_y$$

Expressed in $rtz$-component form:

$$(F_{\text{net}})_r = \sum F_r = ma_r = \frac{mv^2}{r} = m\omega^2 r$$

$$(F_{\text{net}})_t = \sum F_t = \begin{cases} 0 & \text{uniform circular motion} \\ ma_t & \text{nonuniform circular motion} \end{cases}$$

$$(F_{\text{net}})_z = \sum F_z = 0$$
General Principles

Uniform Circular Motion

- \( v \) is constant.
- \( \vec{F}_{\text{net}} \) points toward the center of the circle.
- The centripetal acceleration \( \vec{a} \) points toward the center of the circle. It changes the particle’s direction but not its speed.
Nonuniform Circular Motion

- $\nu$ changes.
- $\vec{a}$ is parallel to $\vec{F}_{net}$.
- The radial component $a_r$ changes the particle’s direction.
- The tangential component $a_t$ changes the particle’s speed.
Important Concepts

\textit{rtz}-coordinates
Important Concepts

**Angular velocity**

\[ \omega = \frac{d\theta}{dt} \]
\[ v_t = \omega r \]

**Angular acceleration**

\[ \alpha = \frac{d\omega}{dt} \]
\[ a_t = \alpha r \]
Applications

Orbits

A circular orbit has radius $r$ if

$$v = \sqrt{rg}$$
Applications

Loops

Circular motion requires a net force pointing to the center. \( n \) must be > 0 for the object to be in contact with a surface.
Chapter 8. Clicker Questions
This acceleration will cause the particle to

A. slow down and curve downward.
B. slow down and curve upward.
C. speed up and curve downward.
D. speed up and curve upward.
E. move to the right and down.
This acceleration will cause the particle to

- **A.** slow down and curve downward.
- B. slow down and curve upward.
- C. speed up and curve downward.
- D. speed up and curve upward.
- E. move to the right and down.
Rank in order, from largest to smallest, the centripetal accelerations \((a_r)_a\) to \((a_r)_e\) of particles a to e.

A. \((a_r)_b > (a_r)_e > (a_r)_a > (a_r)_d > (a_r)_c\)
B. \((a_r)_b > (a_r)_e > (a_r)_a = (a_r)_c > (a_r)_d\)
C. \((a_r)_b = (a_r)_e > (a_r)_a = (a_r)_c > (a_r)_d\)
D. \((a_r)_b > (a_r)_a = (a_r)_c = (a_r)_e > (a_r)_d\)
E. \((a_r)_b > (a_r)_a = (a_r)_a > (a_r)_e > (a_r)_d\)
Rank in order, from largest to smallest, the centripetal accelerations \( (a_r)_a \) to \( (a_r)_e \) of particles a to e.

A. \( (a_r)_b > (a_r)_e > (a_r)_a > (a_r)_d > (a_r)_c \)

B. \( (a_r)_b > (a_r)_e > (a_r)_a = (a_r)_c > (a_r)_d \)  

C. \( (a_r)_b = (a_r)_e > (a_r)_a = (a_r)_c > (a_r)_d \)

D. \( (a_r)_b > (a_r)_a = (a_r)_c = (a_r)_e > (a_r)_d \)

E. \( (a_r)_b > (a_r)_a = (a_r)_a > (a_r)_e > (a_r)_d \)
A block on a string spins in a horizontal circle on a frictionless table. Rank order, from largest to smallest, the tensions \( T_a \) to \( T_e \) acting on blocks a to e.

A. \( T_b > T_a > T_d > T_c > T_e \)
B. \( T_e > T_c = T_d > T_a = T_b \)
C. \( T_e > T_d > T_c > T_b > T_a \)
D. \( T_d > T_b = T_e > T_c > T_a \)
E. \( T_d > T_b > T_e > T_c > T_a \)
A block on a string spins in a horizontal circle on a frictionless table. Rank order, from largest to smallest, the tensions $T_a$ to $T_e$ acting on blocks a to e.

A. $T_b > T_a > T_d > T_c > T_e$
B. $T_e > T_c = T_d > T_a = T_b$
C. $T_e > T_d > T_c > T_b > T_a$  
D. $T_d > T_b = T_e > T_c > T_a$
E. $T_d > T_b > T_e > T_c > T_a$
A car is rolling over the top of a hill at speed $v$. At this instant,

A. $n > w$.
B. $n < w$.
C. $n = w$.
D. We can’t tell about $n$ without knowing $v$. 
A car is rolling over the top of a hill at speed $v$. At this instant,

A. $n > w$.

B. $n < w$. **(Correct)**

C. $n = w$.

D. We can’t tell about $n$ without knowing $v$.  

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A ball on a string is swung in a vertical circle. The string happens to break when it is parallel to the ground and the ball is moving up. Which trajectory does the ball follow?
A ball on a string is swung in a vertical circle. The string happens to break when it is parallel to the ground and the ball is moving up. Which trajectory does the ball follow?